## §5.2 Derivatives of Logarithmic and Trigonometric Functions

Proposition 5.4 (Derivative of inverse function):
Let $I \subset \mathbb{R}$ be a non-trivial Interval, $f: I \longrightarrow \mathbb{R}$ a continuous, strictly monotonic increasing function and $g = f^{-1}: J \longrightarrow \mathbb{R}$ the corresponding inverse function ($J = f(I)$).
If $f$ is differentiable at $x \in I$ and $f'(x) \neq 0$, then $g$ is differentiable at $y := f(x)$ and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}$$

Proof:
Let $(y_k)_{k \in \mathbb{N}} \subset J \setminus \{y\}$ (so $y_k \neq y$) a sequence with $\lim_{k \to \infty} y_k = y$. Set $x_k := g(y_k)$.
As $g$ is continuous (Prop. 4.6), we have

$$\lim_{k \to \infty} x_k = x, \quad x_k \neq x \ \forall k \ (g \text{ is bijective})$$

Thus we compute

$$\lim_{k\to\infty} \frac{g(y_k)-g(y)}{y_k-y} = \lim_{k\to\infty} \frac{x_k-x}{f(x_k)-f(x)}$$

$$= \lim_{k\to\infty} \frac{1}{\frac{f(x_k)-f(x)}{x_k-x}} = \frac{1}{f'(x)}$$

Therefore $g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}$

$\square$

<u>Example 5.6:</u>

$\log : \mathbb{R}_{>0} \to \mathbb{R}$ is the inverse function to $\exp : \mathbb{R} \to \mathbb{R}$. Thus Prop. 5.4 gives:

$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{\exp(\log x)} = \frac{1}{x}$$

<u>Remark 5.3:</u>

We have the following expression for the number $e$:

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

<u>Proof:</u>

As $\log'(1) = 1$, we have

$$\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

Now use

$$\left(1+\frac{1}{n}\right)^n = \exp\left(n \log\left(1+\frac{1}{n}\right)\right),$$

and therefore due to continuity of exp:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \exp(1) = e$$

$\square$

In order to compute derivatives of sin, cos and other trigonometric functions, we need to work with complex numbers briefly.

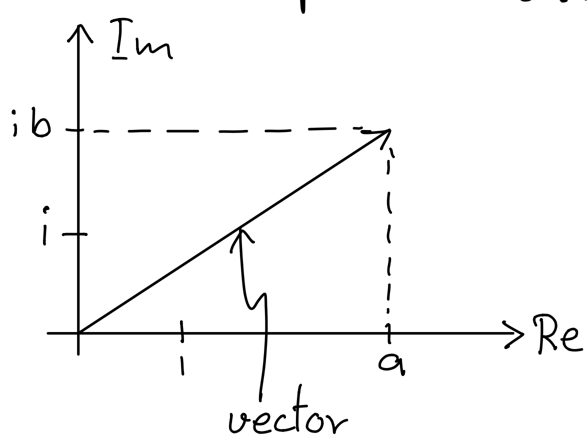
<u>Recall</u> (see Lecture 1):

A complex number $z \in \mathbb{C}$ can be written as:

$$z = a + ib, \quad a, b \in \mathbb{R}$$

and we have $i^2 = -1$

Geometric representation:




sum:

$(a+ib)+(c+id)$
$= a+c+i(b+d)$

product:

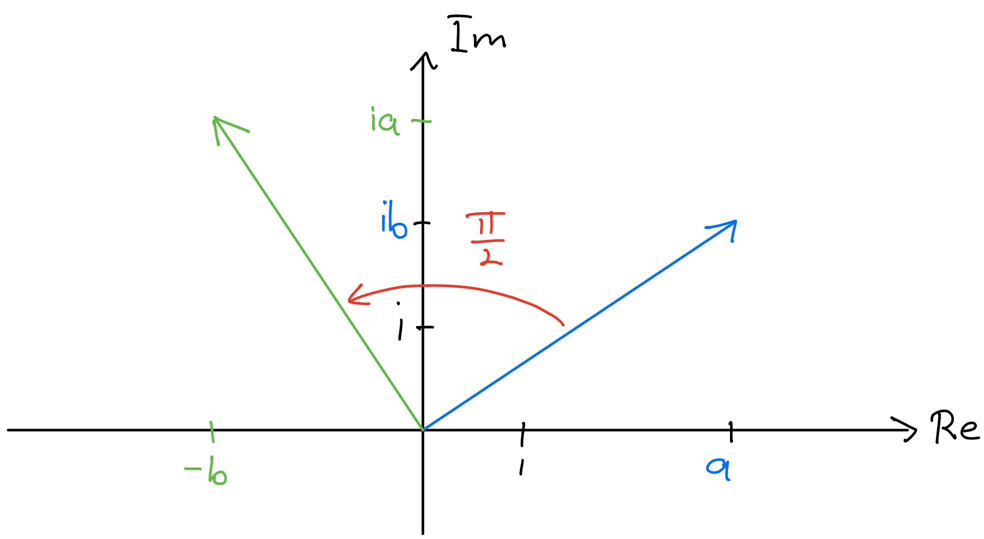
$(a+ib)(c+id)$
$= ac-bd+i(bc+ad)$

Absolute value: $|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$

where $\bar{z}$ denotes the "complex conjugate" of $z$: $\bar{z} := a - ib$

multiplication by $i$:

$$i(a+ib) = -b + ia$$




$\Rightarrow$ We see that multiplication by $i$ amounts to rotation by $\frac{\pi}{2}$ degrees!

Now let's look at the following function:

$$f(t) = e^{it}$$

We have $f(0) = e^0 = 1$ and

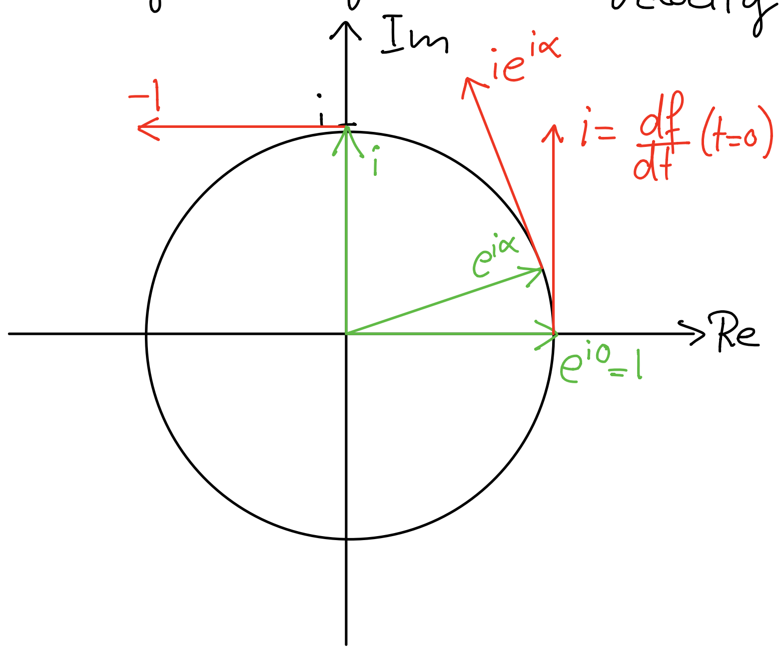
$$|f(t)|^2 = e^{it}\overline{e^{it}} = e^{it}e^{-it} = e^0 = 1$$

Furthermore,

$$\frac{d}{dt} f(t) = ie^{it} \implies \left|\frac{df}{dt}\right| = i(-i)\, e^{it} e^{-it} = 1$$

velocity

Pictorially, this gives

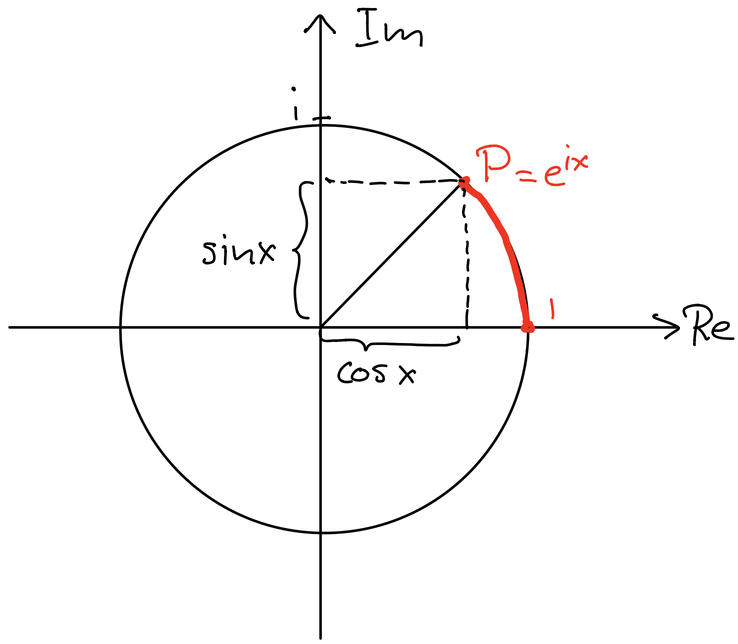



We can think of $f(t)$ as describing the motion of a particle on the plane:

- $|f(t)| = 1 \implies$ the particle moves along the unit circle
- $\left|\frac{df}{dt}\right| = 1 \implies$ the particle traverses the full circle in a time $\Delta t = 2\pi$

Altogether we see that $e^{it}$ is a map from $[0, 2\pi)$ to the unit circle on $\mathbb{C}$.

Let us next look at an arbitrary point $P$ of the unit circle $\Rightarrow \exists\ x \in [0, 2\pi)$ with $P = e^{ix}$




This gives the famous "Euler formula":

$$e^{ix} = \cos x + i \sin x$$

Proposition 5.5:

For all $x \in \mathbb{R}$ we have:

i) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

ii) $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$

iii) $\cos^2 x + \sin^2 x = 1$

Proof:

These follow directly from Euler's formula. $\square$

## Proposition 5.6:

The functions $\cos : \mathbb{R} \to \mathbb{R}$ and $\sin : \mathbb{R} \to \mathbb{R}$ are continuous on all of $\mathbb{R}$.

**Proof:**

Let $a \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ a sequence with $\lim_{n\to\infty} x_n = a$. Then we have $\lim_{n\to\infty} e^{ix_n} = e^{ia}$ and therefore

$$\lim_{n\to\infty} \cos x_n = \lim_{n\to\infty} \operatorname{Re}(e^{ix_n}) = \operatorname{Re}(e^{ia}) = \cos a,$$

$$\lim_{n\to\infty} \sin x_n = \lim \operatorname{Im}(e^{ix_n}) = \operatorname{Im}(e^{ia}) = \sin a.$$

$\Longrightarrow$ cos and sin are continuous in $a$. $\square$

## Proposition 5.7:

For all $x, y \in \mathbb{R}$ we have

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

and in particular:

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1,$$

$$\sin(2x) = 2 \sin x \cos x$$

**Proof:**

Use $e^{i(x+y)} = e^{ix + iy} = e^{ix} e^{iy}$

$\Rightarrow$ Together with the Euler formula this gives:

$$\cos(x+y) + i\sin(x+y) = (\cos x + i\sin x)(\cos y + i\sin y)$$
$$= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y)$$

Now compare real and imaginary parts and the statement follows. $\square$

We are now ready for

<u>Proposition 5.8:</u>

$$\cos'(x) = -\sin x, \quad \sin'(x) = \cos x$$

<u>Proof:</u>

We have that $\frac{d}{dx} e^{ix} = ie^{ix}$. Plugging this into the Euler formula, we get

$$\frac{d}{dx}(\cos x + i\sin x) = i(\cos x + i\sin x) = -\sin x + i\cos x$$

Comparing real and imaginary parts of both sides gives the statement. $\square$